Quantum Information: Solutions to Exercises

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Preface

The following pages present outline, or sketch solutions, to the exercises in my book *Quantum Information*, published by Oxford University Press in June 2009. The solutions are arranged into eight 'chapters' to match the chapters in the book. Equation numbers refer to the correspondingly numbered equations in the book.

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Errata

Shannon taught us that all communications channels contain errors and, inevitably, *Quantum Information* is no exception. Following is a list of corrections that I have come across since the initial publication.

p. 74 There is an error in Fig. 3.7. The final Jones matrix in the table should be

$$\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} .$$

p. 112 In exercise (4.5) "density" should be "density operator"

p. 98 There is an $\hbar$ missing in eqn 4.36, which should be

$$|x_m, p_m\rangle = (2\pi\sigma^2)^{-1/4} \int dx \, \exp\left[-\frac{(x - x_m)^2}{4\sigma^2} + i\frac{p_m x}{\hbar} \right] .$$

Also on this page, there a normalization factor missing in eqn 4.39, which should be

$$\mathcal{P}(x_m) = (2\pi\sigma^2)^{-1/2} \int dx \, \langle x|\hat{\rho}|x\rangle \exp\left[-\frac{(x - x_m)^2}{2\sigma^2} \right] .$$

p. 113 In exercise (4.24) the operator $\hat{\Upsilon}$ should be defined as

$$\hat{\Upsilon} = \sum_{j=1}^{M} \hat{\pi}_j \hat{w}_j .$$

p. 135 In eqn 5.63, there is an error in the normalization of the wavefunction. The prefactor should be $(\pi\sigma^2)^{-1/2}$

p. 162 In exercise (6.23) there is an error in the transformations given They should be:

$$\hat{U}_{\text{swap}}^{1/2}|00\rangle = |00\rangle$$

$$\hat{U}^{1/2}_{\text{swap}}|01\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}(|01\rangle + i|10\rangle)$$
$$\hat{U}^{1/2}_{\text{swap}}|10\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}(|10\rangle + i|01\rangle)$$
$$\hat{U}^{1/2}_{\text{swap}}|11\rangle = |11\rangle$$

p.155 Figure 6.26 has the incorrect powers on the final X gates. The correct from of the figure is given here in Fig. **??**.

p. 194 In exercise (7.20) the condition for the two strings to have the same parity should be $\bar{b} \cdot c + b \cdot \bar{c} = 0 \bmod 2$.

p. 228 In exercise (8.21) the inequality is quoted the wrong way around. It should be

$$S(C) + S(B) \leq S(AB) + S(AC)\,.$$

p. 230 In exercise (8.44) the two signal states should be

$$\hat{\rho}_1 = q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$$
$$\hat{\rho}_2 = (1-q)|0\rangle\langle 0| + q|1\rangle\langle 1|\,.$$

Probability and information

1

(1.1) We know that the joint and single-event probabilities are related by

$$P(a_i) = \sum_j P(a_i, b_j)$$

and that the joint probabilities are necessarily greater than or equal to zero. It follows immediately that

$$P(a_i) \geq P(a_i, b_j)$$

with equality occuring only if

$$\begin{aligned} P(a_i, b_k) &= 0 \qquad k \neq 0 \\ \Rightarrow P(a_i, b_k) &= P(a_i)\delta_{jk} \,. \end{aligned}$$

(1.2) No it does not. The conditional probabilities are related by Bayes' theorem and are not in general equal.

(1.3) We can read off from the probability tree the probabilities

$$P(a_1) = \frac{1}{2} \qquad P(a_2) = \frac{1}{3} \qquad P(a_3) = \frac{1}{6}$$

and the conditional probabilities

$$\begin{array}{lll} P(b_1|a_1) = \frac{1}{4} & P(b_2|a_1) = \frac{1}{4} & P(b_3|a_1) = \frac{1}{2} \\ P(b_1|a_2) = \frac{2}{3} & P(b_2|a_2) = \frac{1}{3} & P(b_3|a_2) = 0 \\ P(b_1|a_3) = \frac{1}{3} & P(b_2|a_3) = \frac{1}{3} & P(b_3|a_3) = \frac{1}{3} \,. \end{array}$$

It is now straightforward to construct the joint probabilities $P(a_i|b_j)$ and then extract the required probabilities:

$$P(b_1) = \frac{29}{72} \qquad P(b_2) = \frac{7}{24} \qquad P(b_3) = \frac{11}{36}$$

and conditional probabilities:

$$\begin{array}{lll} P(a_1|b_1) = \frac{2}{29} & P(a_2|b_1) = \frac{16}{29} & P(a_3|b_1) = \frac{4}{29} \\ P(a_1|b_2) = \frac{3}{7} & P(a_2|b_2) = \frac{8}{21} & P(a_3|b_2) = \frac{4}{21} \\ P(a_1|b_3) = \frac{9}{11} & P(a_2|b_3) = 0 & P(a_3|b_3) = \frac{2}{11} \,. \end{array}$$

From this we can readily read off the entries to construct the required probability tree.

(1.4) The problem gives the probabilities for arriving on time given that the long and shirt routes are taken:

$$P(b_{\mathrm{O}}|a_{\mathrm{s}}) = 1 \qquad P(b_{\mathrm{O}}|a_{\mathrm{l}}) = \frac{3}{4}$$

and probabilities for taking the long and short routes:

$$P(a_{\mathrm{l}}) = \frac{1}{4} \qquad P(a_{\mathrm{s}}) = \frac{3}{4}\,.$$

From the conditional probabilities we have

$$P(b_{\mathrm{L}}|a_{\mathrm{s}}) = 1 - P(b_{\mathrm{O}}|a_{\mathrm{s}}) = 0$$
$$P(b_{\mathrm{L}}|a_{\mathrm{l}}) = 1 - P(b_{\mathrm{O}}|a_{\mathrm{l}}) = \frac{1}{4}\,.$$

The required conditional probabilities are, therefore,

$$P(a_{\mathrm{l}}|b_{\mathrm{L}}) = \frac{P(b_{\mathrm{L}}|a_{\mathrm{l}})P(a_{\mathrm{l}})}{P(b_{\mathrm{L}}|a_{\mathrm{l}})P(a_{\mathrm{l}}) + P(b_{\mathrm{L}}|a_{\mathrm{s}})P(a_{\mathrm{s}})} = 1$$
$$P(a_{\mathrm{s}}|b_{\mathrm{L}}) = 1 - P(a_{\mathrm{l}}|b_{\mathrm{L}}) = 0\,.$$

(1.5) Each particle is detected with probability η and missed with probability $1-\eta$. This means that if we detect n given that N were present, then we also miss $N-n$ particles. There are $\frac{N!}{n!(N-n)!}$ was for this to happen and hence $P(n|N) = \frac{N!}{n!(N-n)!}\eta^n(1-\eta)^{N-n}$. We solve for $P(N|n)$ using Bayes' theorem:

$$P(N|n) = \frac{P(n|N)P(N)}{P(n)}\,.$$

In each case we need to find $P(n)$ using

$$P(n) = \sum_{n=0}^{\infty} P(n|N)P(N)\,.$$

(i) For the Poisson distribution we find

$$P(n) = \sum_{N=n}^{\infty} \frac{N!}{n!(N-n)!}\eta^n(1-\eta)^{N-n}e^{-\bar{N}}\frac{\bar{N}^N}{N!}$$
$$= e^{-\eta\bar{N}}\frac{(\eta\bar{N})^n}{n!}\,.$$

Note that this is a Poisson distribution with mean $\eta\bar{N}$. Bayes' theorem gives the answer

$$P(N|n) = e^{-\bar{N}(1-\eta)}\frac{[\bar{N}(1-\eta)]^{N-n}}{(N-n)!}\,.$$

(ii) The case in which all $P(N)$ are equal needs careful handling as each of the probabilities is zero! We can overcome this problem by use fo a suitable limit. Let the probabilities, $P(N)$, for $N = 0, 1, \cdots, s$ each be $1/(s+1)$ and the probabilities for $N > s$ be zero. We shall take the limit $s \to \infty$ at the end of the calculation. We start by calculating $P(n)$:

$$P(n) = \sum_{N=n}^{s} \frac{N!}{n!(N-n)!} \eta^n (1-\eta)^{N-n} \frac{1}{s+1} .$$

Hence

$$P(N|n) = \frac{\frac{N!}{n!(N-n)!} \eta^n (1-\eta)^{N-n} \frac{1}{s+1}}{\sum_{N'=n}^{\infty} \frac{N'!}{n!(N'-n)!} \eta^n (1-\eta)^{N'-n} \frac{1}{s+1}} .$$

Taking the limit as $s \to \infty$ then gives

$$P(N|n) = \frac{N!}{n!(N-n)!} \eta^{n+1} (1-\eta)^{N-n} .$$

(iii) The problem here is that we need the probability distribution $P(N)$ but we know only that the mean number is $\bar{N}$:

$$\sum_{N=0}^{\infty} N P(N) = \bar{N} .$$

In the absence of any further information we use Jaynes's Max Ent method to maximize

$$H_e = -\sum_N P(N) \ln P(N)$$

subject to the constraints that the probabilities sum to unity and give the correct value for $\bar{N}$. It is straightforward to use Lagrange's method of undertermined multipliers to get

$$P(N) = \frac{\bar{N}^N}{(\bar{N}+1)^{N+1}} .$$

We can now calculate $P(n)$ and hence the required conditional probability:

$$\begin{aligned} P(n) &= \sum_{N'=n}^{\infty} \frac{N'!}{n!(N'-n)!} \eta^n (1-\eta)^{N'-n} \frac{\bar{N}^{N'}}{(\bar{N}+1)^{N'+1}} \\ &= \frac{(\eta\bar{N})^n}{(1+\eta\bar{N})^{n+1}} \\ \Rightarrow P(N|n) &= \frac{N!}{n!(N-n)!} [(1-\eta)\bar{N}]^{N-n} \frac{(1+\eta\bar{N})^{n+1}}{(\bar{N}+1)^{N+1}} . \end{aligned}$$

It is interesting to compare this result with that from part (ii). In the limit $\bar{N} \to \infty$, the conditional probabilities become equal, as they should.

(1.6)

$$\begin{aligned} P(a_i, b_j, c_k) &= P(a_i|b_j, c_k)P(b_j, c_k) \\ &= P(c_k|a_i, b_j)P(a_i, b_j) \\ \Rightarrow P(a_i|b_j, c_k) &= \frac{P(c_k|a_i, b_j)P(a_i, b_j)}{P(b_j, c_k)} \,. \end{aligned}$$

(1.7) We start from eqn 1.21:

$$\begin{aligned} P(a_i|b_j) &= \ell(a_i|b_j)P(a_i) \\ \Rightarrow \ell(a_i|b_j) &= \frac{P(a_i|b_j)}{P(a_i)} = \frac{P(a_i, b_j)}{P(a_i)P(b_j)} \,, \end{aligned}$$

which is manifestly symmetric in a_i and b_j.

(1.8) The prior probabilities are

$$P(BB) = \frac{1}{3} \qquad P(Bb) = \frac{2}{3} \,,$$

and the likelihoods are

$$\ell(BB|x_i = \text{black}) = 1 \qquad \ell(Bb|x_i = \text{black}) = 0 \,.$$

It then follows that

$$\begin{aligned} P(BB|x_i = \text{black}) &\propto 1^n \frac{1}{3} \\ P(Bb|x_i = \text{black}) &\propto \left(\frac{1}{2}\right)^n \frac{2}{3} \,. \end{aligned}$$

Normalizing then gives

$$\begin{aligned} P(BB|x_i = \text{black}) &= \frac{1^n \frac{1}{3}}{1^n \frac{1}{3} + \left(\frac{1}{2}\right)^n \frac{2}{3}} \\ &= 1 - \frac{1}{2^{n-1} + 1} \,. \end{aligned}$$

(1.9) The probabilities that each of the players wins in the first round are

$$P(\text{Amy}) = \frac{1}{2} \qquad P(\text{Barbara}) = \frac{1}{4} \qquad P(\text{Claire}) = \frac{1}{8} \,.$$

The probability that each wins on the second round are 1/8 multiplied by these same probbabilities. In each round the probabilities are in the ration 4:2:1. It follows that the total probabilities that each player will win are

$$P(\text{Amy}) = \frac{4}{7} \qquad P(\text{Barbara}) = \frac{2}{7} \qquad P(\text{Claire}) = \frac{1}{7} \,.$$

(1.10) This is the famous Monty Hall problem. Keeping the originally chosen box does not take into account the additional information provided by displaying an empty box. You should change to the remaining unopen box and will then win with probability 2/3. If this seems strange, then consider that another way of stating the problem is for the host to offer you to keep your one box or take *both* the remaining boxes. There is much scope here for classroom discussion!

(1.11)

(a) The possibilities left are (Boy, Boy), (Boy, Girl) and (Girl, Boy) and all are equally probable. Hence the probability that they are both boys is 1/3.

(b) The possibilities are (Reuben, Boy not Reuben). (Boy not Reuben, Reuben), (Reuben, Girl), (Girl, Reuben) and (Reuben, Reuben). The probability that they are both boys is

$$\begin{aligned} P(\mathrm{BB}) &= \frac{P(\mathrm{R},\mathrm{B}(\not{\mathrm{R}})) + P(\mathrm{B}(\not{\mathrm{R}}),\mathrm{R}) + P(\mathrm{R},\mathrm{R})}{P(\mathrm{R},\mathrm{B}(\not{\mathrm{R}})) + P(\mathrm{B}(\not{\mathrm{R}}),\mathrm{R}) + P(\mathrm{R},\mathrm{R}) + P(\mathrm{R},\mathrm{G}) + P(\mathrm{G},\mathrm{R})} \\ &= \frac{2P(\mathrm{R},\mathrm{B}) - P(\mathrm{R},\mathrm{R})}{2P(\mathrm{R},\mathrm{B}) - P(\mathrm{R},\mathrm{R}) + 2P(\mathrm{R},\mathrm{G})} \\ &= \frac{P(R) - P(\mathrm{R},\mathrm{R})}{2P(R) - P(\mathrm{R},\mathrm{R})} \\ &= \frac{1 - P(R)}{2 - P(R)} \\ &\approx \frac{1}{2}\,, \end{aligned}$$

where the last line follows because Reuben is an unusual name. This problem (and other teasers) is treated in more depth in the book by Mlodinow.

(1.12) The strategy is for each player to look at the other players cards and to make a guess or to decline to guess on the basis of what he/she sees. If a player sees that the colours of the other two players cards are different then he/she declines to guess. If he/she sees that they are both red (black) then he/she bets that he/she has a black (red) card. This strategy wins if there are two cards of the same colour and one that is different. It loses if all three cards are the same colour. Hence the winning probability is $\frac{3}{4}$.

It is true that any individual player will guess the colour of his/her card correctly with probability $\frac{1}{2}$, but this strategy ensures that when a player guesses correctly they are the *only* player making a guess, but when they guess incorrectly *all* the players are guessing. It works by correlating the guesses.

(1.13) It follows from the definition of the logarithm that

$$x = a^{\log_a x} = b^{\log_b x}\,.$$

Taking the logarithm in base b gives

$$\log_b x = \log_b a \log_a x\,,$$